A shallow-liquid theory in magnetohydrodynamics

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The non-linear and linear 'shallow-water' theories, which describe long gravity waves on the free surface of an inviscid liquid, are extended to the case of an electrically conducting liquid on a horizontal bottom, in the presence of a vertical magnetic field. The dish holding the liquid, and the medium outside it, are assumed to be non-conducting. The approximate equations are based on a small ratio of depth to wavelength, on the properties of mercury, and on a moderate magnetic field strength. These equations have a 'magneto-hydraulic' character, for in the shallow liquid layer the horizontal fluid velocity and current density are independent of the vertical co-ordinate.

Some explicit solutions of the linear equations are obtained for plane flows and for axi-symmetric flows in which the velocity vector lies in a vertical, meridional plane. The amplitudes of waves in a dish, and the amplitudes behind wave fronts progressing into undisturbed liquid, are found to be exponentially damped, the mechanical energy associated with a disturbance being dissipated by Joule heating.

The approximate non-linear equations for plane flow are studied by means of characteristic variables, and it appears that, because of the magnetic damping effect, there is less qualitative difference between solutions of the non-linear and linear approximate equations at large times than is the case when the magnetic field is absent. In particular, the characteristic curves depart only a finite distance from their 'undisturbed positions'.

1. Introduction

At the beginning of a general article on the magnetohydrodynamics of liquids, Lehnert (1952) describes some simple experiments made with mercury in a glass dish 20 cm in diameter and about 4 cm in depth. He points out that the surface waves generated by moving a peg in the liquid, or by tapping the dish, 'disappear' when a vertical magnetic field is imposed, and that whereas the liquid appears to have a 'water-like consistence' when there is no field, it 'acts as a thick syrup' when the field is applied.

In the present paper an attempt is made to describe such phenomena theoretically: to this end, stringent approximations have of course to be made. We examine the effect of a vertical magnetic field on long gravity waves in a shallow, inviscid, electrically conducting liquid. The problem is felt to be of

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interest because its relative simplicity offers some hope of understanding: the fluid is incompressible and, in the case of mercury, has a substantial and uniform conductivity; the theory for zero magnetic field is well established (Lamb 1932; Stoker 1957); and experiments can be made without excessive difficulty.

The paper is in three parts. In the first, approximate governing equations are found by an order-of-magnitude analysis which is based on a small ratio of depth to wavelength, on the properties of mercury, and on a moderate field strength. As in the case of no magnetic field, two sets of equations are found: a non-linear set, appropriate when the wave amplitude and the depth of the liquid layer are comparable, and a linear set, appropriate when the amplitude is of even smaller order than the depth. With the derivation of these approximations, a simplified physical picture also emerges. To the lowest order, the horizontal components of velocity, electric field, and current density, and the vertical component of the magnetic field, are constant across the liquid layer (as this is traversed vertically); the remaining components vary linearly. To the scale of the field outside, where the material is assumed non-conducting, the liquid appears as a current sheet.

In the second part of the paper, a number of solutions of the linear equations are found; these describe, for plane flow and axial symmetry,† the motion of the free surface in a dish (in terms of modes or standing waves), and the motion due to travelling waves in a layer which is unbounded laterally. In the third part, 'non-linear effects' in plane flow are studied: although even the approximate, non-linear equations are still too difficult to be solved exactly, we obtain some idea of the nature of their solutions by using characteristic co-ordinates.

Theories related to our problem have, of course, been developed by other writers. Lundquist (1952) gave, among much else, a shallow-liquid theory based on infinite electrical conductivity. As he himself observed, and as will be seen below, this assumption is inappropriate to experiments on the laboratory scale. Stewartson (1957) developed a theory to describe some later experiments of Lehnert, and studied the steady viscous flow of mercury resting on a plane, an inner disk of which rotated, while the outer part remained at rest: the magnetic field was assumed to be zero or large.

It has been remarked above that the ordinary non-linear and linear shallow-water theories (Lamb 1932, §§ 187 and 169, respectively, or Stoker 1957, §§ 2.2 and 10.13) are 'well established'. It is certainly true that both theories have a long history, but they appear to remain a subject of dispute among workers of far greater authority than the present writer (see, for example, Stoker 1957, p. 342; Ursell 1953; and Longuet-Higgins 1958). Since these theories form the starting-point of the present paper, however, some discussion of their validity is essential. The relevant parameters are ϵ, the ratio of mean depth to wavelength, and δ, the ratio of amplitude to mean depth. There seems little doubt that if ϵ, measured from the shortest wave in some finite space-time domain, is small (breaking waves, bores, and hydraulic jumps being thus excluded), and δ is $O(1)$, the non-linear theory is a valid first approximation in that domain. If ϵ and δ are both small, the linear shallow-water theory is a valid first approximation. We

† 'Axial symmetry' here means not only symmetry about a (vertical) axis, but also that the velocity vector lies in a meridian plane.

further conclude, from the sources cited above,† that if ϵ and δ are both small, the non-linear approximation is an improvement on the linear one only if ϵ^2/δ is also small, since only then are the additional terms in the approximate non-linear equations larger than the terms neglected in both cases. (If ϵ^2/δ is $O(1)$ or large, the linear shallow-water theory is the more consistent approximation, and improvements upon it take a form different from that of the non-linear shallow-water theory.) There remains the question of bores and breaking waves. In the present paper we follow Lamb (1932, § 187) and Stoker (1957, § 10.6) in incorporating these in the framework of both approximate theories, but we regard this as an empirical procedure, in the nature of mathematical hydraulics, which provides a crude over-all description of the phenomena.

PART I. THE APPROXIMATE EQUATIONS

2. The non-linear approximation

We write Maxwell's and the hydrodynamic equations in terms of the electric field strength **E**, the magnetic induction **B** (which we also call the magnetic field strength), the current density **J**, and the fluid velocity **V**; T denotes time, P fluid pressure, ρ density of the liquid, ω charge density, κ dielectric constant, σ electrical conductivity, and μ magnetic permeability. Rationalized M.K.S.Q. units are used. The unit vector $\mathbf{i}_3$ points vertically upwards.

Neglecting displacement and convection currents, and the viscosity of the liquid, we have

$$\nabla \wedge \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial T}, \qquad \nabla . \mathbf{E} = \frac{\omega}{\kappa}, \qquad (2.1), (2.2)$$

$$\nabla \wedge \mathbf{B} = \mu \mathbf{J} = \mu\sigma(\mathbf{E} + \mathbf{V} \wedge \mathbf{B}), \quad \nabla . \mathbf{B} = 0, \qquad (2.3), (2.4)$$

$$\rho \frac{D\mathbf{V}}{DT} = -\nabla P - \mathbf{i}_3 \rho g + \mathbf{J} \wedge \mathbf{B}, \quad \nabla . \mathbf{V} = 0. \qquad (2.5), (2.6)$$

Let (X_1, X_2, X_3) be Cartesian co-ordinates, and consider a flow of conducting liquid bounded by a solid surface $X_3 = 0$, a free surface $X_3 = H(X_1, X_2, T)$, and the vertical wall $\mathscr{D}$ of a dish (which may be at infinity). A vertical magnetic field B_0 is applied, and for simplicity we suppose that this extends throughout space. The medium outside the liquid is assumed to be non-conducting and free of charge (although there may be weak surface charges on the interfaces). The initial disturbance is to be confined to some finite region. The boundary conditions are then as follows.

As $\mathbf{X} \to \infty$, $\quad \mathbf{B} \to \mathbf{i}_3 B_0, \quad \mathbf{E} \to 0;$ (2.7 *a*, *b*)

on $X_3 = 0$, $\quad V_3 = 0;$ (2.8)

on $X_3 = H$, $\quad P = 0, \quad \dfrac{\partial H}{\partial T} + V_1 \dfrac{\partial H}{\partial X_1} + V_2 \dfrac{\partial H}{\partial X_2} - V_3 = 0.$ (2.9 *a*, *b*)

Also, the normal component of velocity must vanish on $\mathscr{D}$; and the vector **B** and the tangential components of **E** must be continuous across the interfaces $X_3 = 0$, H and $\mathscr{D}$, continuity of the tangential components of **B** following from the

† Dr Ursell has also been kind enough to show the present writer how the main conclusions of his 1953 paper may be derived much more simply.

requirement of finite current density when the conductivity is finite. (Here we have assumed that the magnetic permeability, μ, is constant throughout the field. If μ is assigned different values in the liquid and outside it, the tangential components of $\mathbf{B}/\mu$ must be continuous: the corresponding changes, in the theory which follows, are easily made.)

We now introduce the small parameter $\epsilon = H_0/L$, where, in the initial conditions, H_0 is a typical depth and L a length characteristic of changes in the horizontal directions, and seek the leading term in the asymptotic expansion of each dependent variable in powers of $\epsilon^{\frac{1}{2}}$. The other parameters of the problem are the magnetic Reynolds number N, and a dimensionless field strength K:

$$N = \mu\sigma L\sqrt{(gL)}, \quad K = \frac{B_0^2\sigma L}{\rho\sqrt{(gL)}}.$$

For mercury $N \doteqdot 4L^{\frac{3}{2}}$ (L in metres): it is therefore assumed that for laboratory experiments $N \leqslant O(1)$. The order of K will be so chosen that the ordinary theory ($B_0^2\sigma = 0$) falls within the range of our approximation: in the momentum equation the magnetic force is not to dominate the gradient of hydrostatic pressure. We therefore assign to the hydrodynamic variables and to the natural time the orders which apply when $B_0^2\sigma = 0$ (see Stoker 1957, § 2.4). Dimensionless variables of $O(1)$ are introduced by

$$\left.\begin{aligned} &\mathbf{X} = (Lx_1, Lx_2, \epsilon^{\alpha}Lx_3), \quad T = \epsilon^{-\frac{1}{2}}\frac{L}{\sqrt{(gL)}}t, \\ &\mathbf{V} = (\epsilon^{\frac{1}{2}}\sqrt{(gL)}\,v_1, \quad \epsilon^{\frac{1}{2}}\sqrt{(gL)}\,v_2, \quad \epsilon^{\frac{3}{2}}\sqrt{(gL)}\,v_3), \quad P = \epsilon\rho gLp, \quad H = \epsilon Lh, \end{aligned}\right\} \tag{2.10}$$

where α is a symbol defined to be 1 inside the liquid layer and 0 outside it. The orders of the electromagnetic variables are determined from the governing equations in the Appendix, and are found to be those in the following scheme:

$$\left.\begin{aligned} &\mathbf{E} = (\epsilon^{\frac{1}{2}}B_0\sqrt{(gL)}\,e_1, \quad \epsilon^{\frac{1}{2}}B_0\sqrt{(gL)}\,e_2, \quad \epsilon^{\alpha+\frac{1}{2}}B_0\sqrt{(gL)}\,e_3), \\ &\mathbf{B} = (\epsilon^{\frac{3}{2}}NB_0b_1, \quad \epsilon^{\frac{3}{2}}NB_0b_2, \quad B_0+\epsilon^{\frac{3}{2}}NB_0b_3), \\ &\mathbf{J} = \left(\epsilon^{\frac{1}{2}}\frac{NB_0}{\mu L}j_1, \quad \epsilon^{\frac{1}{2}}\frac{NB_0}{\mu L}j_2, \quad \epsilon^{\frac{3}{2}}\frac{NB_0}{\mu L}j_3\right). \end{aligned}\right\} \tag{2.11}$$

It is also found necessary to restrict K to be $O(\epsilon^{\frac{1}{2}})$: in place of K we introduce

$$k = \frac{K}{2\epsilon^{\frac{1}{2}}} = \frac{B_0^2\sigma L}{2\rho\sqrt{(gH_0)}} \sim O(1).$$

For mercury and with $\epsilon = 0{\cdot}1$, the following field strengths correspond to $k = 1$:

L (metres)	0·05	0·10	0·15
B_0 (10^4 gauss)	0·35	0·29	0·26

Note that the lowest-order electric field, which is $O(\epsilon^{\frac{1}{2}})$, is not induced by time variations of the magnetic field, which are $O(\epsilon^2)$. As may be seen from the Appendix point (v), and from what follows, the role of this electric field is to conserve the current within the liquid.

The horizontal components of (2.1), (2.3) become

$$-\frac{\partial e_2}{\partial x_3} = 0, \quad \frac{\partial e_1}{\partial x_3} = 0, \tag{2.12a,b}$$

$$-\frac{\partial b_2}{\partial x_3} = j_1 = e_1 + v_2, \quad \frac{\partial b_1}{\partial x_3} = j_2 = e_2 - v_1; \tag{2.13a,b}$$

and the vertical momentum equation reduces, as usual, to

$$0 = -\frac{\partial p}{\partial x_3} - 1, \quad \text{so that} \quad p = h - x_3,$$

where the boundary condition (2.9a) has been used. The approximate equations which govern the motion of the liquid are then (explanations follow):

$$\frac{\partial h}{\partial t} + \frac{\partial (hv_1)}{\partial x_1} + \frac{\partial (hv_2)}{\partial x_2} = 0, \tag{2.14}$$

$$\frac{Dv_1}{Dt} = -\frac{\partial h}{\partial x_1} + 2k(e_2 - v_1), \quad \frac{Dv_2}{Dt} = -\frac{\partial h}{\partial x_2} - 2k(e_1 + v_2), \tag{2.15a,b}$$

$$\frac{\partial e_2}{\partial x_1} - \frac{\partial e_1}{\partial x_2} = 0, \tag{2.16}$$

$$\frac{\partial}{\partial x_1}\{(e_1 + v_2) h\} + \frac{\partial}{\partial x_2}\{(e_2 - v_1) h\} = 0. \tag{2.17}$$

In (2.15) the pressure gradient and the **e**-terms are independent of x_3 (see (2.12)), and it is assumed that initially v_1 and v_2 are independent of x_3. Then we may write (2.15) in Lagrangian co-ordinates and integrate with respect to time to show that v_1 and v_2 are independent of x_3 for all time. The continuity equation $\nabla . \mathbf{V} = 0$ can now be integrated across the liquid layer to yield (2.14) when the boundary conditions on V_3 are applied. Equation (2.16) is the reduced form of the vertical component of (2.1). Equation (2.17) expresses the conservation of total current within the liquid layer, and may also be derived as follows. In view of the anti-symmetry with respect to x_3 of the outside solution for b_1 and b_2 (see below), and the continuity of these quantities across the interfaces, we may integrate (2.13) to

$$b_1 = (e_2 - v_1)(x_3 - \tfrac{1}{2}h), \quad b_2 = -(e_1 + v_2)(x_3 - \tfrac{1}{2}h); \tag{2.18a,b}$$

and

$$j_3 = e_3 = \frac{\partial b_2}{\partial x_1} - \frac{\partial b_1}{\partial x_2} \tag{2.19}$$

in the liquid. Now the continuity of **B** across $X_3 = 0, H$, and $\nabla \wedge \mathbf{B} = 0$ outside, require $\mathbf{n} . \mathbf{J} = 0$ on $X_3 = 0, H$, and this condition yields (2.17) on both surfaces.

Equations (2.14) to (2.17) must be solved with appropriate initial conditions and with boundary conditions on $\mathscr{D}$. If suffices n and t denote directions normal and tangential to $\mathscr{D}$, such that '$nt3$' form a right-handed system, we require that

$$v_n = 0, \quad e_n + v_t = 0 \quad \text{on} \quad \mathscr{D}.$$

To complete the picture of conditions in the liquid we observe that (2.4) reduces to $\partial b_3/\partial x_3 = 0$; b_3 is thus to be found from the outside solution.

In those special cases (§§ 4, 5) where the dimensionless electric field strength is of higher order than the velocity, the magnetic force in the momentum equation is proportional to the velocity, and is precisely like the quasi-viscous force introduced by Rayleigh (see Lamb 1932, § 242) in his study of the waves caused by application of pressure to the surface of a stream.

Although to our approximation the conduction current is divergence-free, the electric field strength is not: in fact to the lowest order

$$\nabla . \mathbf{E} = -\mathbf{B}_0 . (\nabla \wedge \mathbf{V}),$$

and we shall also have E_3 discontinuous across $X_3 = 0, H$. However, the corresponding charges are $O(\kappa)$—or, in dimensionless form, $O(gL/c^2)$, where c is the velocity of light—and the divergence of conduction current required to create these charges is therefore also $O(\kappa)$, and this is negligible to our approximation.

We now turn to the external problem. To the natural scale L of the outside field the liquid is simply a current sheet, since the boundary conditions on $x_3 = 0-, \epsilon h+$ may be satisfied on $x_3 = 0 \mp$. The field equations reduce to

$$\nabla \wedge \mathbf{b} = 0, \quad \nabla . \mathbf{b} = 0,$$

so that

$$\left.\begin{aligned} \mathbf{b} &= \frac{1}{4\pi}\iint \frac{\mathbf{j}^*(\mathbf{x}') \wedge \mathbf{R}}{R^3}\, dx_1'\, dx_2', \\ \text{where}\quad \mathbf{j}^* &= (j_1 h, j_2 h, 0), \quad \mathbf{R} = \mathbf{i}_1(x_1 - x_1') + \mathbf{i}_2(x_2 - x_2') + \mathbf{i}_3 x_3. \end{aligned}\right\} \tag{2.20}$$

To determine the $\mathbf{E}$ field we have to solve another potential problem

$$\nabla \wedge \mathbf{e} = 0, \quad \nabla . \mathbf{e} = 0;$$

here even boundary values of e_1 and e_2 are given on $x_3 = 0 \mp$, and in general we have to solve an integral equation with a source-type kernel.

We shall make no attempt to obtain explicit solutions for these external fields, but shall concentrate on the behaviour of the liquid.

In the ordinary ($k = 0$) non-linear approximation, as expounded by Stoker, the error is a factor $\{1 + O(\epsilon^2)\}$. In our case, inspection of the neglected terms suggests that the error in the hydrodynamic variables is a factor $\{1 + O(\epsilon)\}$. It is true that in the vertical component of (2.3) we retain terms of $O(\epsilon^{\frac{3}{2}})$ and neglect the $\mathbf{V} \wedge \mathbf{B}$ term, which is $O(\epsilon^2)$, but the corresponding neglected force in the momentum equation is only $O(\epsilon^4)$, compared to $O(\epsilon)$ for the lowest-order magnetic force. Moreover, for axi-symmetric and plane flows (§ 5), $\mathbf{V}$ and $\mathbf{B}$ are in the same vertical plane and $\mathbf{V} \wedge \mathbf{B}$ has no vertical component, so that the ratio of neglected to retained terms is $O(\epsilon)$ in all equations.

3. 'Shock waves'; the energy balance

In principle, bores, hydraulic jumps, or shocks (henceforth we shall use the last term) violate the assumptions of the theory, since in their neighbourhood the vertical acceleration is certainly not negligible; but since shocks, idealized to surfaces of discontinuity, have been fitted with some success into the theory for $k = 0$, we shall attempt to incorporate them here also. This is essentially an empirical step. If the governing equations are written, by means of Gauss's and

Stokes's theorems, in an integral form appropriate to a moving 'control surface',† and then applied in the usual way to a small surface enclosing a portion of the shock, the jump conditions are found to be:

(conservation of mass) $$[h(v_n - w_n)] = 0, \tag{3.1}$$

(momentum) $$[\tfrac{1}{2}h^2 + hv_n(v_n - w_n)] = 0, \quad [v_t] = 0, \tag{3.2a, b}$$

(irrotationality of e_1 and e_2) $$[e_t] = 0, \tag{3.3}$$

(conservation of current) $$[h(e_n + v_t)] = 0, \tag{3.4}$$

where horizontal components normal and tangential to the shock are denoted by suffices n and t; '$nt3$' form a right-handed system; w_n denotes the shock velocity; and square brackets denote, for the moment, a jump or discontinuity operator. The hydrodynamic jump conditions are unchanged by the presence of the magnetic field; as usual, energy considerations (below) require that particles cross the shock from a region of smaller to one of larger depth.

The approximate equations (2.12) *et seq.* are not uniformly valid. In their derivation, we have neglected a number of terms which, if retained, would lead to wave-speeds other than $(gH)^{\frac{1}{2}}$, which is the only characteristic speed implicit in our equations. In the case of a wave travelling into still liquid, these neglected effects would propagate disturbance (of a higher order in ϵ) ahead of the wave-front predicted by our theory. The approximate equations give warning of this when the wave-front is a shock: then the tangential current $j_t h$ is discontinuous, so that b_3 is logarithmically singular (cf. a discontinuity in loading in thin aerofoil theory).

We proceed to the energy balance of our approximate system. First, we must write the appropriate Poynting vector, which is, in the liquid,

$$\left.\begin{aligned} \mathbf{S} &= \epsilon^{\frac{1}{2}} \frac{B_0 \sqrt{(gL)}}{\mu} \mathbf{e} \wedge \mathbf{B}_0 + \epsilon^2 B_0^2 \sigma g L^2 \mathbf{s}, \\ \text{where} \qquad \mathbf{s} &= \mathbf{i}_1 e_2 b_3 - \mathbf{i}_2 e_1 b_3 - \mathbf{i}_3 (e_1 v_2 - e_2 v_1 + e_1^2 + e_2^2)(x_3 - \tfrac{1}{2}h). \end{aligned}\right\} \tag{3.5}$$

Since $\nabla . (\mathbf{e} \wedge \mathbf{B}_0) = \mathbf{B}_0 . (\nabla \wedge \mathbf{e}) = 0$, the integral of $\mathbf{e} \wedge \mathbf{B}_0$ over a closed surface vanishes. Also, s_1 and s_2 are not significant because their flux is across small surfaces.

Consider now a moving volume of liquid, bounded by a vertical cylinder C, of cross-section A, and containing shocks along a number of vertical surfaces of which C_w is typical. (C_w is not necessarily a closed curve in a plane x_3 = constant.) On C, $\mathbf{n}$ is the outward normal; on C_w, $\mathbf{n}$ points in the direction of the relative fluid velocity, $v_n - w_n > 0$. Suffices + and − refer to values on the downstream and upstream sides of C_w, respectively, and we write $\{h(v_n - w_n)\}_+ = \{h(v_n - w_n)\}_- = m$. To construct the energy equation, we perform the differentiation in

$$\frac{d}{dt} \iint_A (\tfrac{1}{2}h^2 + \tfrac{1}{2}v^2 h)\, dA,$$

† The momentum equation must first be multiplied by h. The integral form of (2.16) is $\int_C \mathbf{e} . d\mathbf{l} = 0$, where $d\mathbf{l}$ is a line element of the horizontal circuit C; the fact that C may be partly in, and partly out of, the liquid causes no difficulty since e_1 and e_2 pass continuously through the liquid and do not change significantly with X_3 in any interval of width $O(\epsilon L)$.

and substitute for $\partial h/\partial t$ from the continuity equation and for $\partial v/\partial t$ from the momentum equation. The result is simplified by introduction of the Poynting vector and the shock equations, and is finally found to be

$$-\int_C \tfrac{1}{2}h^2 v_n\,dl = \frac{d}{dt}\iint_A (\tfrac{1}{2}h^2+\tfrac{1}{2}v^2h)\,dA + \Sigma\int_{C_w} \frac{m(h_+-h_-)^3}{4h_+h_-}\,dl$$
$$+2k\iint_A (j^2h + s_3|_{x_3=0}^{h})\,dA. \quad (3.6)$$

Here the left-hand side represents the rate at which the pressures on C are doing work on the volume: on the right there appear the rates of increase of the potential and kinetic energy, of energy dissipation by shocks, of energy dissipation by Joule heating, and of efflux of electromagnetic energy to the external field.

Consider a disturbance initially confined to a finite domain, and choose C outside the wave-front: then the left-hand side of (3.6) vanishes. If we assume that some (or all) of the initial potential and kinetic energy of the disturbance is ultimately dissipated, it is clear that in the ordinary case ($k = 0$) shocks provide the only mechanism for this: in our case other means are available. It therefore seems likely that certain, initially continuous flows, which lead to shocks when $k = 0$, may remain continuous when $k > 0$. In part III it will be shown that this is, in fact, the case.

4. The linear approximation

It has been implicitly assumed so far that the wave amplitude $|H-H_0|_{\max}$ is comparable with the mean depth H_0. We now suppose that the initial conditions introduce a second small parameter

$$\frac{|H-H_0|_{\max}}{H_0} = \delta.$$

Whereas orders with respect to ϵ were written explicitly in (2.10), (2.11), those with respect to δ will be treated implicitly. Like $\partial h/\partial x_1$, the dimensionless velocities v_1, v_2 are now $O(\delta)$, and we write

$$h = 1+\eta(x_1, x_2, t),$$

where η is $O(\delta)$.

In the momentum equation, Dv_1/Dt now reduces to $\partial v_1/\partial t$, and (2.17) reduces to

$$\frac{\partial}{\partial x_1}(e_1+v_2)+\frac{\partial}{\partial x_2}(e_2-v_1) = 0, \quad (4.1)$$

so that the magnetic force in the momentum equation is irrotational, and derivable from a potential. Hence if the vector (v_1, v_2) is irrotational initially (as we assume), it remains so for all time; on the linear theory this is true whether the motion is continuous or not. Then by (2.16) and (4.1)

$$\frac{\partial e_2}{\partial x_1}-\frac{\partial e_1}{\partial x_2} = 0, \quad \frac{\partial e_1}{\partial x_1}+\frac{\partial e_2}{\partial x_2} = 0.$$

If the liquid is unbounded laterally, e_1 and e_2 vanish at infinity, are continuous through the field (by the linearized form of the shock equations), and have no singularities. Hence $e_1 = e_2 = 0$ (although there will be an electric field to a higher order). Again for axi-symmetric flow, and for certain plane flows (§ 5), the electric field vanishes from the hydrodynamic equations, which are then

$$\frac{\partial \eta}{\partial t}+\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2} = 0, \tag{4.2}$$

$$\frac{\partial v_1}{\partial t} = -\frac{\partial \eta}{\partial x_1}-2kv_1, \quad \frac{\partial v_2}{\partial t} = -\frac{\partial \eta}{\partial x_2}-2kv_2. \tag{4.3a, b}$$

If the velocity potential ϕ is introduced, such that

$$v_1 = \frac{\partial \phi}{\partial x_1}, \quad v_2 = \frac{\partial \phi}{\partial x_2}, \quad \eta = -\left(\frac{\partial}{\partial t}+2k\right)\phi, \tag{4.4}$$

the momentum equations are satisfied identically, and the continuity equation becomes

$$\frac{\partial^2 \phi}{\partial x_1^2}+\frac{\partial^2 \phi}{\partial x_2^2} = \frac{\partial^2 \phi}{\partial t^2}+2k\frac{\partial \phi}{\partial t}, \tag{4.5}$$

which is a form of the telegraph equation.

To obtain the energy balance of the linear approximation (with e_1 and e_2 not necessarily vanishing) we rewrite (3.6) for a fixed vertical cylinder C^* of cross-section A^*; subtract the continuity equation

$$-\int_{C^*} hv_n\,dl = \frac{d}{dt}\iint_{A^*} h\,dA,$$

and, writing $h = 1+\eta$, neglect terms of $O(\delta^3)$. There results

$$-\int_{C^*} \eta v_n\,dl = \frac{d}{dt}\iint_{A^*}(\tfrac{1}{2}\eta^2+\tfrac{1}{2}v^2)\,dA+2k\iint_{A^*}(j^2+s_3|_{x_3=0}^{1})\,dA. \tag{4.6}$$

This equation can also be found directly from the linear equations. We observe again that the potential and kinetic energy is continuously dissipated by Joule heating: if C^* is chosen such that $v_n = 0$ there, and if also $e_1 = e_2 = 0$, this is the only effect on the energy within C^*.

5. Plane and axi-symmetric flows

We discuss the plane and axi-symmetric cases together; for this, it is convenient to introduce the symbol β, which is 0 for the former case and 1 for the latter. For the axi-symmetric case, (x_1, x_2, x_3) are identified with cylindrical co-ordinates (r, θ, z), dimensionless as in (2.10). The problems to be considered are those for which $\partial(\ \ )/\partial X_2 = 0$, and $V_2 = B_2 = 0$ initially. It then follows from the full equations, (2.1) to (2.6), and from the boundary conditions, that everywhere and for all time

$$V_2 = B_2 = E_1 = E_3 = 0.$$

Currents flow only in the 2-direction.

We now find from (2.1) that, inside and outside the liquid,

$$\frac{\partial e_2}{\partial x_3} = 0, \quad \frac{1}{x_1^\beta}\frac{\partial}{\partial x_1}(x_1^\beta e_2) = 0, \tag{5.1}$$

and if the boundary condition at infinity is used, $e_2 = 0$ everywhere.

But at this stage we must inquire whether these conditions can be simulated in the laboratory. There are no special difficulties associated with the axi-symmetric case (for which $e_2 = 0$ is valid), but we cannot achieve a perfectly 'plane' state of affairs with flow in a channel (even when viscosity is neglected) because the current must flow in closed loops. Probably the best way of obtaining approximately plane flow in a channel is to make the side walls (that is, the walls parallel to $X_2 = 0$) far apart relative to the wavelength L and of highly conducting material, as Hartmann did in his experiments (1937) on plane Poiseuille flow. Even then the electromagnetic field is 'plane' only in a limited region about the central part of the channel, the boundary condition at infinity cannot be invoked, and we must accept an e_2 which is constant in space over the region in question. Its strength is determined by the condition

$$\iint j_2\,dx_1\,dx_3 = \int (e_2 - v_1)\,h\,dx_1 = 0 \tag{5.2}$$

for the non-linear theory, with $h \sim 1$ when we linearize. However, this still leads to $e_2 = 0$ if (i) the initial conditions are such that h is an even and v_1 an odd function of x_1, or (ii) the x_1-length of the liquid is much greater than the x_1-length of that part of it which is moving, or (iii) $v_1 h$ is oscillatory such that

$$\int v_1 h\,dx_1 \ll \int h\,dx_1.$$

Again, if the side walls are short-circuited by means of a very good solid conductor, parallel to the x_2-axis and outside the liquid, the electric field e_2 will vanish, while the condition

$$\iint j_2\,dx_1\,dx_3 = 0$$

is still satisfied. The contribution to the magnetic field of the current in the solid conductor must, of course, be taken into account.

In what follows we consider only axi-symmetric flows and those plane flows for which the electric field vanishes from the lowest-order hydrodynamic equations. If, as in the axi-symmetric case, the current $\sigma\mathbf{V}\wedge\mathbf{B}$ is exactly parallel to all the insulating boundaries and flows in closed circuits, the electric field is, to the lowest order,

$$\mathbf{E} = (0, \epsilon^2 B_0 \surd(gL)\, e_2^*, 0),$$

where e_2^* is calculated from the time variation of the magnetic perturbations:

$$\frac{\partial e_2^*}{\partial x_3} = N\frac{\partial b_1}{\partial t}, \quad \frac{1}{x_1^\beta}\frac{\partial}{\partial x_1}(x_1^\beta e_2^*) = -N\frac{\partial b_3}{\partial t}.$$

To the lowest order the current depends only on the velocity, and the magnetic perturbations are found from it, as before.

Part II. Some Linearized Solutions for Plane and Axi-symmetric Flow

6. An initial-value problem

We ask what happens to a given initial elevation $f(x_1)$ of the free surface. More precisely, we consider the following problem for plane flow (x_1, x_2, x_3 being

Cartesian co-ordinates, $\beta = 0$), and axi-symmetric flow (x_1, x_2, x_3 being cylindrical co-ordinates, $\beta = 1$):

$$\frac{1}{x_1^\beta}\frac{\partial}{\partial x_1}\left(x_1^\beta \frac{\partial \phi}{\partial x_1}\right) = \frac{\partial^2 \phi}{\partial t^2} + 2k\frac{\partial \phi}{\partial t}; \tag{6.1}$$

at time $t = 0$,

$$\left.\begin{aligned} \eta = -\frac{\partial \phi}{\partial t} - 2k\phi = f(x_1), \quad \frac{\partial \eta}{\partial t} = 0, \\ v_1 = \frac{\partial \phi}{\partial x_1} = 0, \quad \frac{\partial v_1}{\partial t} = -f'(x_1); \end{aligned}\right\} \tag{6.2}$$

at $|x_1| = a$,

$$v_1 = \frac{\partial \eta}{\partial x_1} = 0. \tag{6.3}$$

Waves in a dish of semi-width or radius L ($a = 1$), and in an unbounded fluid ($a \to \infty$) will be considered. In place of x_1 and v_1, we shall now write x and u (when $\beta = 0$) or r and u (when $\beta = 1$).

7. Standing waves in a dish

Here, solutions of two rather elementary eigenvalue problems are required: the method is obvious, and no details of it will be given. In the plane case, to satisfy the requirement of zero total current normal to a flow plane, we specify a disturbance symmetrical about $x = 0$:

$$f(x) = \sum_1^\infty A_n \cos n\pi x \quad (|x| \leqslant 1),$$

where the $n = 0$ term is absent because H_0 is taken to be the mean value of $H(X, 0)$. Then, with $(n^2\pi^2 - k^2)^{\frac{1}{2}} = \omega_n$,

$$\phi = -e^{-kt}\sum_1^\infty A_n \cos n\pi x\, \omega_n^{-1} \sin \omega_n t, \tag{7.1a}$$

$$\eta = e^{-kt}\sum_1^\infty A_n \cos n\pi x \{\cos \omega_n t + k\omega_n^{-1} \sin \omega_n t\}, \tag{7.1b}$$

$$u = e^{-kt}\sum_1^\infty A_n \sin n\pi x\, n\pi \omega_n^{-1} \sin \omega_n t, \tag{7.1c}$$

the trigonometric functions of time becoming hyperbolic functions for $k > n\pi$. The effect of increasing field strength, k, upon any single mode, η_n, is precisely like the damping of a mechanical system: at $k = 0$, η_n oscillates harmonically in time; for $0 < k < n\pi$, the frequency is reduced, and the motion is damped; at $k = n\pi$, η_n moves like $e^{-kt}(1 + kt)$; for $k > n\pi$, η_n behaves like

$$\exp\{[-k + (k^2 - n^2\pi^2)^{\frac{1}{2}}]\,t\};$$

and as $k \to \infty$ (although this is outside the region of validity of our theory), η_n tends not to move.

For the axi-symmetric case, $f(r)$ is expanded in a Dini series (Watson 1944):

$$f(r) = \sum_1^\infty A_n J_0(\lambda_n r) \quad (r \leqslant 1),$$

$$A_n = \frac{2}{J_0^2(\lambda_n)}\int_0^1 f(s)\, J_0(\lambda_n s)\, s\, ds,$$

where λ_n are the positive zeros of J_1, and the $n = 0$ term is absent because H_0 is taken as the mean value of $H(r, 0)$ with respect to horizontal area. Then, with $(\lambda_n^2 - k^2)^{\frac{1}{2}} = \omega_n$,

$$\phi = -e^{-kt} \sum_1^\infty A_n J_0(\lambda_n r)\, \omega_n^{-1} \sin \omega_n t, \tag{7.2a}$$

$$\eta = e^{-kt} \sum_1^\infty A_n J_0(\lambda_n r)\, \{\cos \omega_n t + k\omega_n^{-1} \sin \omega_n t\}, \tag{7.2b}$$

$$u = e^{-kt} \sum_1^\infty A_n J_1(\lambda_n r)\, \lambda_n \omega_n^{-1} \sin \omega_n t. \tag{7.2c}$$

The shape of the modes is slightly different from that in the plane case; their behaviour with time is precisely the same.

8. Travelling waves in an unbounded liquid

We introduce the Laplace transform

$$\bar{F}(x,p) = p \int_0^\infty e^{-pt} F(x,t)\, dt.$$

In the plane case, with the initial conditions $\partial\phi/\partial t = -f(x)$, and $\phi = 0$ at $t = 0$, the differential equation becomes

$$\frac{d^2\bar{\phi}}{dx^2} - (p^2 + 2kp)\, \bar{\phi} = pf(x), \tag{8.1}$$

and we require that $\bar{\phi} \to 0$ as $|x| \to \infty$. Let $(p^2 + 2kp)^{\frac{1}{2}} = q$; there is a cut from $p = -2k$ to $p = 0$, and q denotes that branch which is positive on the positive real axis. The fundamental solution of (8.1) is

$$g(x; \lambda) = -\frac{e^{-q|x-\lambda|}}{2q};$$

using this, and inverting the transform, we find that

$$\phi = -\frac{1}{2\pi i} \int_{\mathscr{L}} \frac{e^{pt}}{2q}\, dp \int_{-\infty}^{\infty} e^{-q|x-\lambda|} f(\lambda)\, d\lambda, \tag{8.2a}$$

$$= -\tfrac{1}{2} e^{-kt} \int_{x-t}^{x+t} I_0(k\{t^2 - (x-\lambda)^2\}^{\frac{1}{2}})\, f(\lambda)\, d\lambda, \tag{8.2b}$$

where $\mathscr{L}$ denotes the Bromwich path ($c - i\infty$ to $c + i\infty$, $c > 0$, in the present case), I_n is the modified Bessel function of the first kind and order n, and the inversion integral has been evaluated by standard methods (see, for example, van der Pol & Bremmer 1955).

Consider first the simple if unrealistic case of an initial elevation concentrated at the origin such that the raised area of liquid is γ: $f(x) = \gamma\delta^*(x)$, where δ^* is the Dirac function. This will give an indication of the behaviour of more sensible solutions. Then, with U denoting the Heaviside unit function,

$$\phi = -\tfrac{1}{2}\gamma\, e^{-kt} I_0(k\{t^2 - x^2\}^{\frac{1}{2}})\, U(t - |x|), \tag{8.3a}$$

$$\eta = \tfrac{1}{2}\gamma\, e^{-kt} \left\{ \delta^*(t - |x|) + \left[\frac{kt}{(t^2 - x^2)^{\frac{1}{2}}} I_1(k\{t^2 - x^2\}^{\frac{1}{2}}) + kI_0(k\{t^2 - x^2\}^{\frac{1}{2}}) \right] U(t - |x|) \right\}, \tag{8.3b}$$

$$u = \tfrac{1}{2}\gamma\, e^{-kt} \left\{ (\operatorname{sgn} x)\, \delta^*(t - |x|) + \frac{kx}{(t^2 - x^2)^{\frac{1}{2}}} I_1(k\{t^2 - x^2\}^{\frac{1}{2}})\, U(t - |x|) \right\}. \tag{8.3c}$$

Comparing this with the ordinary ($k = 0$) solution, $\eta = (\frac{1}{2}\gamma)\,\delta^*(t-|x|)$, we observe that the surface elevation is exponentially damped near the wave-fronts $|x| = t$, where $I_0 \sim 1$, $I_1 \sim (\frac{1}{2}k)\,(t^2-x^2)^{\frac{1}{2}}$, but that there is also a residual wave between the fronts which decays only slowly near $x = 0$, where $I_0 \sim I_1 \sim e^{kt}(2\pi kt)^{-\frac{1}{2}}$.

Such a state of affairs is hardly surprising. We would expect the magnetic force to damp most strongly the high-frequency Fourier components which dominate near the wave-front; on the other hand, continuity requires that $\int_{-\infty}^{\infty} \eta\, dx$ be constant, so that if the elevation is damped near the fronts, it must have compensating values elsewhere.

To examine this effect in more detail, we study the solution due to an initial step, $f(x) = \delta U(-x)$: the flows due to double or multi-steps can be easily constructed from this by superposition. After a little reduction one finds that

$$\eta = \frac{\delta}{2\pi i}\int_{\mathscr{L}} \frac{e^{pt-qx}}{2p}\,dp \quad (x \geqslant 0), \tag{8.4a}$$

$$= \delta - \frac{\delta}{2\pi i}\int_{\mathscr{L}} \frac{e^{pt+qx}}{2p}\,dp \quad (x \leqslant 0), \tag{8.4b}$$

so that $\eta(0,t) = \frac{1}{2}\delta$, and for $x \leqslant 0$, $\eta(x,t) = \delta - \eta(|x|,t)$; hence it is sufficient to study η for $x \geqslant 0$. Also

$$\eta = \frac{\delta\, e^{-kt}}{2}\left(\frac{\partial}{\partial t}+k\right)\int_x^t I_0(k\{t^2-\mu^2\}^{\frac{1}{2}})\,d\mu \quad (0 \leqslant x < t), \tag{8.5}$$

$$u = \frac{\delta\, e^{-kt}}{2}\, I_0(k\{t^2-x^2\}^{\frac{1}{2}}). \tag{8.6}$$

By expanding the integrand of (8.4a) in series for p large, or the integrand of (8.5) for $(t-\mu)$ small, one finds that near the wave-front $x = t$,

$$\eta = \frac{\delta\, e^{-kt}}{2}\left\{1+(kt+2)\,k\beta+(k^2t^2+4kt)\frac{k^2\beta^2}{4}+(k^3t^3+6k^2t^2-6kt-24)\frac{k^3\beta^3}{36}+\ldots\right\}, \tag{8.7}$$

where $\beta = \frac{1}{2}(t-x)$. This series may also be rearranged to yield one valid for small times across the whole disturbed region $t > |x|$. By applying the method of steepest descents to (8.4a) one finds that, in a region between the wave-front and $x = 0$, where t, $(t^2-x^2)^{\frac{1}{2}}$, and $t-(t^2-x^2)^{\frac{1}{2}}$ are all large,

$$\eta \sim \frac{\delta}{2(2\pi)^{\frac{1}{2}}}\,\frac{\exp\{-kt+k(t^2-x^2)^{\frac{1}{2}}\}}{kt-k(t^2-x^2)^{\frac{1}{2}}}\,\frac{kx}{\{k^2(t^2-x^2)\}^{\frac{1}{4}}}. \tag{8.8}$$

These results suggest the following approximation (which will be useful in part III):

$$\eta = \frac{\delta\, e^{-kt}}{2}\left[e^{kt}-\left\{\frac{3}{2kt}\,(e^{kt}-1-kt-\tfrac{1}{2}k^2t^2)+1+\tfrac{1}{2}kt\right\}kx+\{e^{kt}-1-kt-\tfrac{1}{2}k^2t^2\}\frac{x^3}{2t^3}\right]. \tag{8.9}$$

Here a cubic wave profile has been fitted to give the correct values at $x = 0$, $|x| = t$, the correct slope at $|x| = t$, and the correct curvature (zero) at $x = 0$. In figure 1 the values given by (8.9) are compared with values obtained from (8.5) by a careful numerical integration. For $kt = 1, 2$, the 'exact' and approximate curves are indistinguishable in the figure. The equations (8.4a) to (8.9), and figure 1, describe the following flow pattern. A compressive discontinuity moves along $x = t$ and an expansive one along $x = -t$ (by 'compressive' we mean that the particles cross from smaller to larger h, by 'expansive', the opposite), the

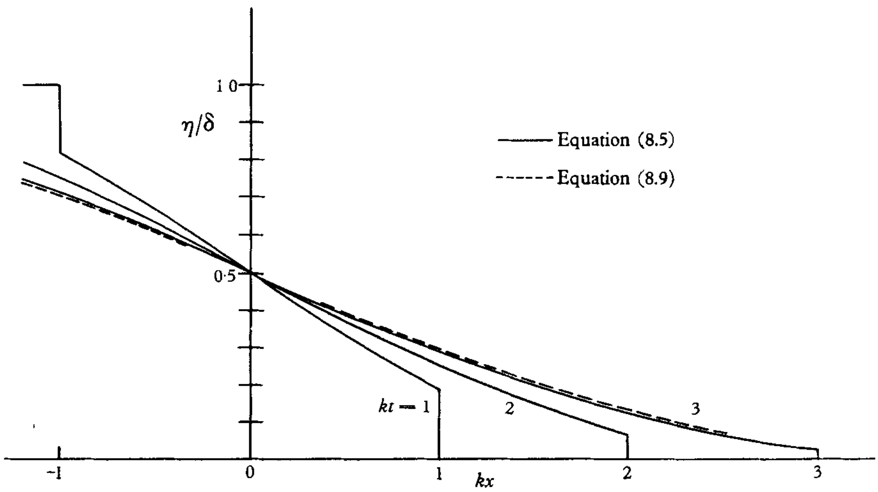


FIGURE 1. Wave profiles due to a step.

heights of both being exponentially damped. The velocity u is positive everywhere, and is similarly damped. Halfway between the fronts, the surface elevation remains $\frac{1}{2}\delta$ and the velocity attains its maximum value, decaying only like $t^{-\frac{1}{2}}$. There is, of course, a continuous transition from the values at the centre to those just inside the wave-fronts.

Figure 2 shows wave profiles resulting from a double step (square wave) $f(x) = \delta U(-x+1) - \delta U(-x-1)$. These pictures, together with equations (8.7) and (8.8), confirm our earlier description of a wave damped at its edges, but tending to maintain a larger elevation near its centre. For the double step, the ordinary theory ($k = 0$) predicts that when $t > 1$, the elevation consists of two portions, in the intervals $t-1 < |x| < t+1$, each of height $\frac{1}{2}\delta$; between them, in $|x| < t-1$, there is no disturbance. In the present case the major part of the elevation remains in $|x| < t-1$.

If the initial surface elevation is sinusoidal everywhere, we recover standing wave solutions like those of § 7, but with arbitrary wave numbers. Thus with $f(x) = \delta\, e^{i\nu x}$, (8.2$a$) yields

$$\phi = -\delta\, e^{i\nu x}(\nu^2 - k^2)^{-\frac{1}{2}}\, e^{-kt} \sin(\{\nu^2 - k^2\}^{\frac{1}{2}} t).$$

Clearly this solution could be made the basis of a more general one by superposition, but we have thought it more interesting and convenient to proceed from the solution for a pulse, which is the basis of (8.2*b*).

In the axi-symmetric problem, the solution corresponding to (8.2*a*) is

$$\phi = -\frac{1}{2\pi i}\int_{\mathscr{L}} e^{pt}\,dp\left\{K_0(qr)\int_0^r I_0(q\lambda)\,f(\lambda)\,\lambda\,d\lambda + I_0(qr)\int_r^\infty K_0(q\lambda)\,f(\lambda)\,\lambda\,d\lambda\right\}, \tag{8.10}$$

where K_0 is the modified Bessel function of the second kind and order zero (Watson 1944), and a cut is required along the entire negative real axis of the p-plane to make K_0 single-valued.

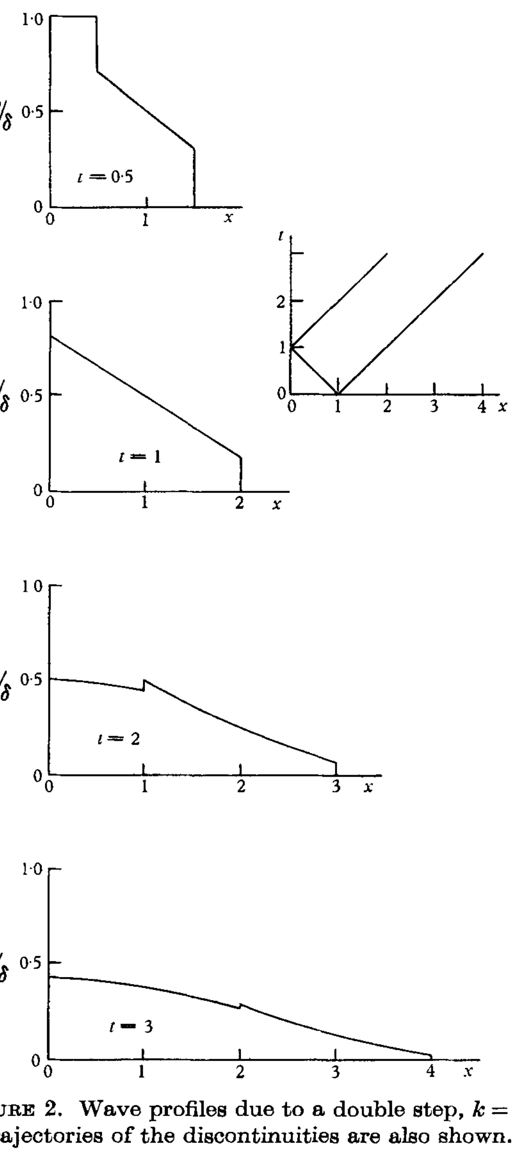


FIGURE 2. Wave profiles due to a double step, $k = 1$. Trajectories of the discontinuities are also shown.

For an initial elevation concentrated at the origin, such that the raised volume is γ, we have $f(r) = \gamma\delta^*(r)/\pi r$, and

$$\phi = -\frac{\gamma}{2\pi}\frac{1}{2\pi i}\int_{\mathscr{L}} e^{pt} K_0(qr)\,dp,$$

$$= -\frac{\gamma}{2\pi}\frac{e^{-kt}\cosh(k\{t^2-r^2\}^{\frac{1}{2}})}{(t^2-r^2)^{\frac{1}{2}}}\,U(t-r). \tag{8.11}$$

This solution shows essentially the same properties as the corresponding one for plane flow, (8.3a); however, there is now an algebraic singularity (as well as a Dirac pulse in η and u) at the wave-front $r = t$, and the decay of elevation near the origin is slightly more rapid than before ($O(t^{-1})$ instead of $O(t^{-\frac{1}{2}})$).

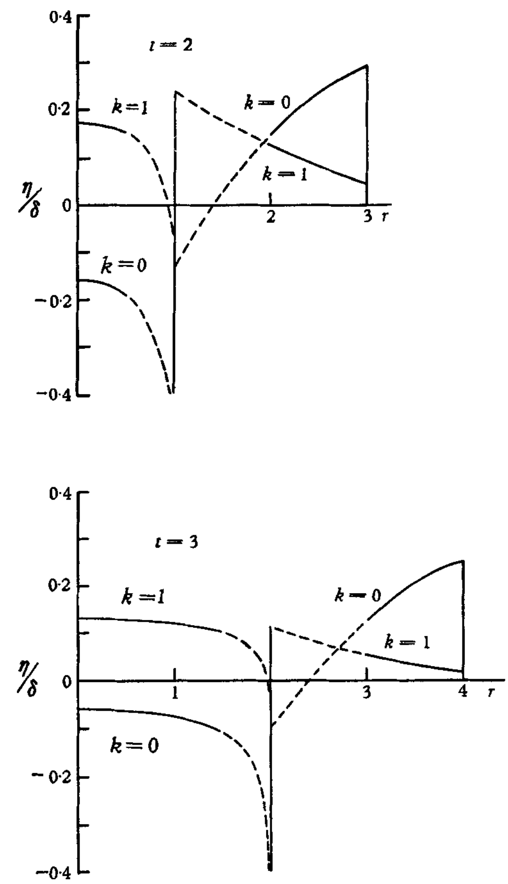


FIGURE 3. Wave profiles due to a cylindrical step.

For a cylindrical step, such that $f(r) = \delta U(1-r)$, the inner integrals of (8.10) may be evaluated, and one can find series for η behind the outgoing ($r = 1+t$) and incoming ($r = 1-t$) wave-fronts. In addition, one can show that immediately behind the reflected wave-front ($r = t-1$)

$$\eta \sim \frac{\delta\,e^{-kt}}{2\pi r^{\frac{1}{2}}}\log(t-r-1), \tag{8.12}$$

and one can find a series for η in powers of r^2 in the domain $r < t-1$. (The logarithmic singularity is a spurious result of the linearization, but suggestive of large disturbances: Meyer (1948) has studied a similar singularity in a related problem.) These results have been used to construct the wave profiles shown in figure 3. Although the whole profiles are shown, the series used, which were truncated after three terms, are believed to be reliable only for the regions shown by solid curves. The remaining parts of the profiles were constructed from consideration of the singularity behind the reflected wave-front, and from the requirement of continuity that for all time

$$\int_0^\infty \eta \, 2\pi r \, dr = \pi\delta.$$

We observe that for $k = 0$ most of the elevation is carried along behind the outer wave-front, that it decreases inwards, and that inside the reflected wave-front the elevation is negative. On the other hand, for $k = 1$ the elevation is damped behind the outer wave-front and increases inwards. The reflected wave-front is again expansive, but the elevation inside it is predominantly positive; as in the plane case, the wave tends to be 'held together' by the magnetic force.

PART III. SOME NON-LINEAR EFFECTS IN PLANE FLOW

9. The approximate equations in characteristic variables

In this final part of the paper, we make attempts to solve the initial-value problem of § 6 for plane flow and a liquid unbounded laterally when the governing field equations are those of our non-linear approximation. Disturbances now move with a variable propagation velocity $(gH)^{\frac{1}{2}}$ relative to the fluid particles (which are themselves moving), and we are particularly interested in the question of whether, according to our approximation, this effect causes shocks to form (waves to break) in fields which are initially continuous. Clearly, there will be two opposing effects: the usual tendency of compressive wavelets to overtake those ahead of them, and thus to form a shock, and the damping effect of the magnetic force on the disturbances immediately behind a wave-front. We also seek to improve the solutions we have obtained for flows whose initial elevation contains steps; where the linearized theory showed compressive and expansive discontinuities moving with speed $(gH_0)^{\frac{1}{2}}$, we try to construct a picture which distinguishes between shocks and centred expansion waves, and which gives an indication of their true velocities. Fortunately, the work of Friedrichs (1948), Lighthill (1949, 1955), Whitham (1952), Lin (1954), and Fox (1955) points the way to the resolution of such questions: we have merely to try to follow, incorporating the effect of the new damping term.

Using suffices to denote partial derivatives, we write the governing equations, namely

$$\left.\begin{aligned} h_t + (uh)_x &= 0, \\ u_t + uu_x + h_x + 2ku &= 0, \end{aligned}\right\} \tag{9.1}$$

in characteristic form:

$$\left.\begin{aligned} &\text{on } C_+ \quad \frac{x_\alpha}{t_\alpha} = u + h^{\frac{1}{2}}, \quad 2(h^{\frac{1}{2}})_\alpha + u_\alpha + 2kut_\alpha = 0, && (9.2a, b) \\ &\text{on } C_- \quad \frac{x_\beta}{t_\beta} = u - h^{\frac{1}{2}}, \quad -2(h^{\frac{1}{2}})_\beta + u_\beta + 2kut_\beta = 0. && (9.3a, b) \end{aligned}\right\}$$

Here α and β are curvilinear co-ordinates which are defined by (9.2a) and (9.3a), and we fix their scale by specifying that on $t = 0$, $\alpha = -\beta = \frac{1}{2}x$. With $k > 0$, (9.2) and (9.3) no longer admit 'simple wave' solutions, and we shall have to be content with series expansions of exact solutions. These will be the simplest possible examples of the following two types: (i) wave-front series, in powers of a characteristic co-ordinate measured from a wave-front across which the normal derivatives of u and h are discontinuous, and (ii) small-disturbance series, in which $h(\alpha,\beta)$, $u(\alpha,\beta)$, $x(\alpha,\beta)$, and $t(\alpha,\beta)$ are expanded in powers of the small parameter δ. In connexion with the latter it may be worth reminding the reader that, while we are perfectly free to construct mathematical solutions of the non-linear approximation in powers of δ, the non-linear theory is a better approximation than the linear one only if $\delta \gg \epsilon^2$.

10. Wave-front series

Let the (dimensionless) elevation of the free surface at time $t = 0$ be

$$\left.\begin{aligned} h(x,0) \equiv 1+f(x) &= 1 \quad (x \geqslant 0), \\ &= 1-Ax+\tfrac{1}{2}Bx^2+\ldots \quad (x \leqslant 0), \end{aligned}\right\} \tag{10.1}$$

the initial velocity being zero. Discontinuities will be propagated along $\alpha = 0$ and $\beta = 0$. In the region $\beta < 0$, $t \geqslant 0$, the variables are undisturbed, $h = 1$, $u = 0$, $x = \alpha - \beta$, $t = \alpha + \beta$; in the region $\alpha > 0$, $\beta > 0$, all four variables are expanded in Taylor series about $\beta = 0$; and in the region $\alpha < 0$, $t \geqslant 0$, they are expanded in double Taylor series about $\alpha = 0$, $\beta = 0$. This reduces the problem to a series of algebraic and linear ordinary differential equations, with boundary conditions on $t = 0$, $\alpha = 0$, and $\beta = 0$. One finds that for $\alpha > 0$, $\beta > 0$,

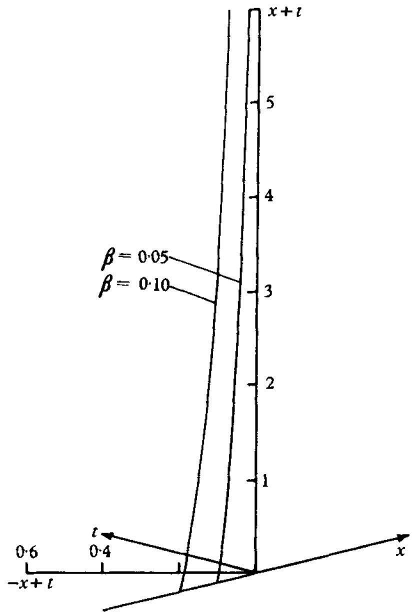
$$\left.\begin{aligned} h &= 1+A\,e^{-k\alpha}\,\beta+\{A^2(-e^{-2k\alpha}-\tfrac{3}{4}k\alpha\,e^{-k\alpha}+\tfrac{1}{2}\,e^{-k\alpha}) \\ &\qquad +Ak^2\alpha\,e^{-k\alpha}+2B\,e^{-k\alpha}\}\,\tfrac{1}{2}\beta^2+\ldots, \\ u &= A\,e^{-k\alpha}\beta+\{A^2(-3\,e^{-2k\alpha}-\tfrac{3}{4}k\alpha\,e^{-k\alpha}+2\,e^{-k\alpha}) \\ &\qquad +Ak(k\alpha\,e^{-k\alpha}-2e^{-k\alpha})+2B\,e^{-k\alpha}\}\,\tfrac{1}{2}\beta^2+\ldots, \\ x &= \alpha-\left\{\frac{3A}{4k}(e^{-k\alpha}-1)+1\right\}\beta+\left\{\frac{A^2}{16k}(27\,e^{-2k\alpha}+9k\alpha\,e^{-k\alpha}-12\,e^{-k\alpha}-15)\right. \\ &\qquad \left.+A(-\tfrac{3}{4}k\alpha\,e^{-k\alpha}+\tfrac{1}{2}\,e^{-k\alpha}+1)-\frac{3B}{2k}(e^{-k\alpha}-1)\right\}\tfrac{1}{2}\beta^2+\ldots, \\ t &= \alpha+\left\{\frac{3A}{4k}(e^{-k\alpha}-1)+1\right\}\beta+\left\{\frac{A^2}{16k}(-21\,e^{-2k\alpha}-9k\alpha\,e^{-k\alpha}+6\,e^{-k\alpha}+15)\right. \\ &\qquad \left.+A(\tfrac{3}{4}k\alpha\,e^{-k\alpha}-1)+\frac{3B}{2k}(e^{-k\alpha}-1)\right\}\tfrac{1}{2}\beta^2+\ldots. \end{aligned}\right\} \tag{10.2}$$

The main justification for including these rather clumsy expressions here is that they give us a glimpse of the non-linear terms (those in A^2) in an exact solution of equations (9.1). We first observe that the function $t-x$ given by (10.2) is bounded, with $k > 0$ and β bounded, for all α, including $\alpha \to \infty$; whereas for $k = 0$, $t-x \sim O(\alpha)$. Thus with $k > 0$ a characteristic $\beta = \beta^*$ can move only a finite distance away from its 'undisturbed position' $t-x = 2\beta^*$, and the tendency of shocks to form is much reduced. In particular, the derivatives

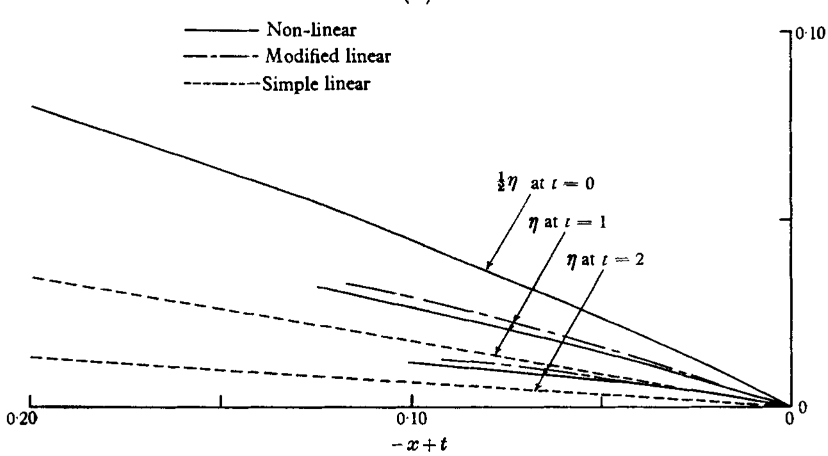
$$-x_\beta(\alpha, 0) = t_\beta(\alpha, 0) = 1-\frac{3A}{4k}(1-e^{-k\alpha})$$

cannot vanish (which means that at the wave-front characteristics cannot converge, and shocks cannot form), if, and only if,

$$\frac{3A}{4k} < 1, \quad \text{i.e. if} \quad \frac{\vartheta\rho}{\sigma B_0^2}\left(\frac{g}{H_0}\right)^{\frac{1}{2}} < \tfrac{2}{3}, \tag{10.3}$$



(a)



(b)

FIGURE 4. (a) The paths of the characteristics $\beta = 0{\cdot}05$, $0{\cdot}10$; (b) the wave profiles near the front of an initially parabolic wave.

where ϑ is the initial wave slope, $-H_X(0-,0)$, and is positive for a compressive wave-front. Next we observe that if A and B are $O(\delta)$, then in any of the expressions of (10.2) the ratio of second-order (A^2) terms to first-order (A, B) terms is $O(\delta)$ *uniformly* (that is, this ratio is $O(1)$ with respect to α and β for all α and β, including $\alpha \to \infty$), both for $k = 0$ and $k > 0$. This suggests that if we expand solutions in the $\alpha\beta$-plane in powers of δ, then, as in the case $k = 0$, we shall obtain a uniform asymptotic expansion.

Of course, equation (10.2) can also be used for approximate calculations of the leading sectors of flow fields. As an example, we take† $A = 1$, $B = -2$, so that the initial wave profile has the shape, near $X = 0$, of the parabola

$$\frac{H}{H_0} = 1 - \frac{X}{L} - \left(\frac{X}{L}\right)^2 \quad \left(-1 \leqslant \frac{X}{L} \leqslant 0\right),$$

of length L and amplitude $\frac{1}{4}H_0$; we also take $k = 1$, so that the condition (10.3) is satisfied. Figure 4(a) shows the characteristics $\beta = 0{\cdot}05$, $0{\cdot}10$, which move forward towards the wave-front, but not sufficiently to form a shock. Figure 4(b) shows the leading part of the wave-profile at various times: the appropriate reference quantity at time $t = 0$ is not η, but $\frac{1}{2}\eta$, because the wave splits into forward and backward moving parts. The profiles plotted are (i) the non-linear result of equation (10.2), (ii) the 'modified linear' result, which can be obtained either from (10.2), by neglecting terms in A^2, or by applying the method of § 11 and expanding for β small, and (iii) the 'simple linear' result, which can be obtained either from (10.2), by setting $\alpha \equiv \frac{1}{2}(x+t)$, $\beta \equiv \frac{1}{2}(-x+t)$ and neglecting terms in A^2 in the expressions for h and u, or from equation (8.2). All these results are, of course, only asymptotic for $\beta \to 0$ and are shown only for $\beta \leqslant 0{\cdot}10$.

The forward movement of the characteristics lessens the decay of wave-slope with time predicted by the simple linear theory, and the modified linear theory of § 11, which takes the shift of characteristics into account, yields much more accurate results than the simple one. Of course, the good agreement between non-linear and modified linear results is to be expected in the present case, since their difference occurs only in the curvature (β^2) terms of (10.2).

11. Small-disturbance series

We now consider series of the form

$$h - 1 = \delta\eta_1(\alpha,\beta) + \ldots, \quad u = \delta u_1(\alpha,\beta) + \ldots, \qquad (11.1\,a, b)$$

$$x = \alpha - \beta + \delta x_1(\alpha,\beta) + \ldots, \quad t = \alpha + \beta + \delta t_1(\alpha,\beta) + \ldots, \qquad (11.2\,a, b)$$

and assume, for reasons which follow, that such expansions are uniformly asymptotic series in the $\alpha\beta$-plane. For the case $k = 0$, in which the series for $h^{\frac{1}{2}}$ and u terminate after the terms of $O(\delta)$, Fox (1955) has shown that the infinite series for x, t are not only uniformly asymptotic, but actually converge for quite substantial values of the initial disturbances: it seems unlikely that the inclusion of a damping term should alter this result. Some slight evidence for our assumption has also been obtained in the previous section. Finally, we refer to the work

† Note that taking $|A| = 1$ merely defines L to be $H_0/|\vartheta|$.

of Whitham (1952), in which a number of results obtained by essentially the same hypothesis are checked against more exact solutions.

Substituting (11.1*a*, *b*) and (11.2*b*) into the 'flow equations' (9.2*b*), (9.3*b*), we find that η_1 and u_1 both satisfy

$$F_{\alpha\beta}+k(F_\alpha+F_\beta) = 0, \tag{11.3}$$

and this is the linearized equation,

$$F_{xx}-F_{tt}-2kF_t = 0,$$

with $x = \alpha-\beta$ and $t = \alpha+\beta$. Accordingly, η_1 and u_1 are simply the linearized solutions of part II with $x+t$ replaced by 2α and $-x+t$ replaced by 2β, but our viewpoint is now quite different. Whereas $x+t$ and $-x+t$ are Cartesian coordinates in the xt-plane, α and β are to be regarded as variables constant on a characteristic: the essential difference lies in the $O(\delta)$ terms of (11.2), which are to be included in the first approximation.

Substituting (11.2*a*, *b*) into the 'direction equations' (9.2*a*), (9.3*a*), we now find that

$$x_{1\alpha}-t_{1\alpha} = \tfrac{1}{2}\eta_1+u_1,$$
$$x_{1\beta}+t_{1\beta} = -\tfrac{1}{2}\eta_1+u_1,$$

so that, in view of the boundary conditions $x_1 = 0$, $t_1 = 0$ on $\beta = -\alpha$,

$$x_1-t_1 = \int_{-\beta}^{\alpha}\{\tfrac{1}{2}\eta_1(\alpha',\beta)+u_1(\alpha',\beta)\}\,d\alpha', \tag{11.4a}$$

$$x_1+t_1 = \int_{-\alpha}^{\beta}\{-\tfrac{1}{2}\eta_1(\alpha,\beta')+u_1(\alpha,\beta')\}\,d\beta'. \tag{11.4b}$$

Since η_1 and u_1 are known, these are, in principle, known functions.

Consider the initial-value problem studied previously: we now write $f(x) \equiv \delta f_1(x)$ for the initial surface elevation. We ask whether shocks can form from an initially continuous surface elevation; that is, whether the Jacobian

$$J \equiv x_\alpha t_\beta - x_\beta t_\alpha = 2+\delta(x_{1\alpha}+t_{1\beta}-x_{1\beta}+t_{1\alpha})+\dots$$

can vanish somewhere in the field. Now it can be shown, by straightforward if lengthy calculations, that if $f_1(x)$ is continuous and its derivative is bounded, such that $|f_1(x)| < M$, $|f_1'(x)| < N$ for all x, then with $\alpha+\beta = \tau\ (\geqslant 0)$,

$$\left.\begin{aligned}|x_{1\alpha}+t_{1\alpha}-x_{1\beta}+t_{1\beta}| &< M(3-2e^{-k\tau})+\frac{3N}{2k}(1-e^{-k\tau})(3-e^{-k\tau}),\\ &\leqslant 3M+\frac{9N}{2k}.\end{aligned}\right\} \tag{11.5}$$

Hence with $k > 0$ the first-order term in the Jacobian remains $O(\delta)$ for all τ (whereas with $k = 0$ this term is $O(\delta\tau)$), and by the assumption of uniformly asymptotic series, the higher-order terms behave similarly. According to the first-order theory and quite crude estimates, therefore, shocks will not occur if

$$\delta\left(\frac{3M}{2}+\frac{9N}{4k}\right) \leqslant 1. \tag{11.6}$$

We proceed to the case where the initial surface elevation is discontinuous: the discontinuities are assumed to be separated by distances greater than $O(\delta)$. Then the linearized solutions η_1 and u_1 are discontinuous across certain characteristics: let $\beta = \beta^*$ be such a characteristic. By (11.4*a*), $\beta = \beta^*-$ and $\beta = \beta^*+$ will be different curves in the xt-plane, and the quantity

$$\delta[x_1 - t_1]^{\beta^*+}_{\beta=\beta^*-} = G(\alpha), \quad \text{say,}$$

is a measure of the distance between them. For shocks G is positive, and the domains $\beta < \beta^*$ and $\beta > \beta^*$ overlap in the xt-plane; for centred expansion waves G is negative, and there is a gap between these two domains in the xt-plane (figure 5). Because the jumps in η and u are exponentially damped, however, the width of the overlapping region or gap remains $O(\delta)$ for all α if $k > 0$, whereas for $k = 0$ it is $O(\delta\alpha)$. It then follows from arguments rather similar to those used in another paper (Fraenkel 1959, § 3) that the shock equations are satisfied, to our order of accuracy, if the shock is drawn halfway between the characteristics $\beta = \beta^*-$ and $\beta = \beta^*+$ in the xt-plane, the solutions between the shock and $\beta = \beta^*-$ on the sheet $\beta < \beta^*$, and between $\beta = \beta^*+$ and the shock on the sheet $\beta > \beta^*$, being discarded. The gap corresponding to an expansion fan may be filled, to our approximation, by taking a linear variation of h and u across it.

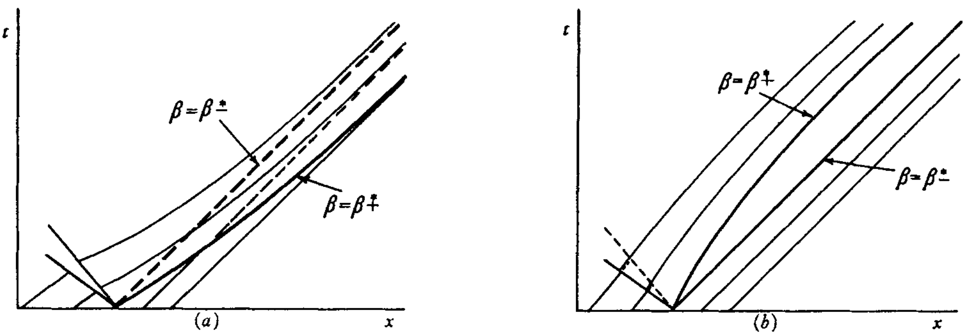


FIGURE 5. (*a*) The 'overlapping region' near a shock, (*b*) the 'gap' produced by a centred expansion wave.

For the initial elevation

$$f(x) = \delta\{1 + O(x)\}\, U(-x),$$

(11.4*a*, *b*) then show, upon elimination of α, that

$$\text{on} \quad \beta = 0-, \; x - t = 0; \quad \text{on} \quad \beta = 0+, \; x - t = \frac{3\delta}{4k}(1 - e^{-kt}) + O(\delta^2). \tag{11.7}$$

Hence for $\delta > 0$ the shock is

$$x - t = \frac{3\delta}{8k}(1 - e^{-kt}) + O(\delta^2), \tag{11.8}$$

while for $\delta < 0$ the equations (11.7) give the boundaries of the expansion fan.

Figure 6 shows the shock and characteristics pattern, according to the present theory, in the problem of a double step considered in part II (with $k = 1$). Here δ has been taken equal to the rather large value 0·4, to make the departure of the characteristics from their undisturbed positions clearly visible. The integrations of (11.4a, b) to find $\beta = 0{\cdot}5-$ were done numerically, the integrands being given by (8.6) and (8.9) with x and t replaced by $\alpha-\beta-1$ and $\alpha+\beta$; the increments for $\beta = 0{\cdot}5+$, and the equations of the other curves, were found analytically from the equations above. Like figure 4(a), the figure illustrates the convection of disturbances implicit in the characteristic equations.

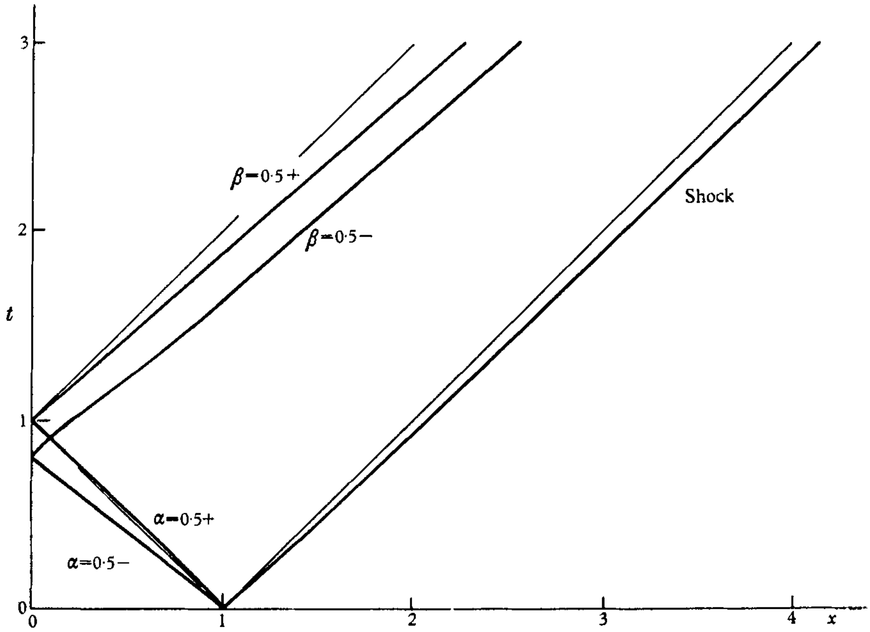


FIGURE 6. Shock and characteristics pattern resulting from a double step; $k = 1$, $\delta = 0{\cdot}4$. The fine lines are the characteristics of the simple linear theory.

Since for $k > 0$ and δ sufficiently small, (i) condition (11.6) is satisfied, so that shocks do not form from initially continuous surface elevations, and (ii) shocks and centred expansion waves depart only a distance $O(\delta)$ from their undisturbed positions, even for large times, we conclude that the linear theory of part II is uniformly valid as an approximation to the non-linear equations for small depth. (These latter, however, are not a uniformly valid approximation to the full equations, as was mentioned in § 3).

12. Concluding remarks

Lehnert's experiments (1952) are only described in a qualitative way in his paper, and a proper comparison with the present theory is therefore not possible. However, the most conspicuous results of the theory, namely the exponential decay of the wave amplitude and fluid velocity in a dish, and of the disturbances immediately behind a wave-front (§§ 7, 8), appear to agree with Lehnert's observation that the waves 'disappear' under the action of a magnetic field. On the other hand, his remark that the fluid 'acts as a thick syrup' in the presence of the field must be treated with caution, for this suggests an effective viscosity, whereas the theory involves a resisting body force proportional to the velocity rather than to its second derivatives.

We have obtained solutions for only the simplest problems, in which the electric field strength is negligible and the current density is simply $\sigma \mathbf{V} \wedge \mathbf{B}_0$ to the lowest order. In this case the most significant effect in the energy balance of the whole flow (equations (3.6) and (4.6)) is the Joule heating, and the decay of disturbances is not surprising. Although solutions have been obtained only for plane and axisymmetric flow, they are believed to be fairly representative of more general situations in a liquid unbounded laterally, for then the electric field still vanishes from the linearized hydrodynamic equations (§ 4).

For general, unsymmetrical waves in a finite dish made of insulating material, the electric field is significant even in the linear approximation, since the normal current must vanish at the vertical wall. In that case the surface elevation still satisfies the equation

$$\frac{\partial^2 \eta}{\partial x_1^2} + \frac{\partial^2 \eta}{\partial x_2^2} = \frac{\partial^2 \eta}{\partial t^2} + 2k \frac{\partial \eta}{\partial t},$$

but the relation of η to the velocity, and the boundary conditions, are less simple than in the cases studied here. In addition, the energy balance involves an exchange of electromagnetic energy between the liquid and the external field. A further investigation of these flows is intended.

There is a certain similarity (first conjectured by H. W. Liepmann) between the wave-front behaviour of the cylindrical waves described in § 8, and that of cylindrical sound pulses propagating into a gas having solid-body rotation and constant temperature (Fraenkel 1959). For zero magnetic field and zero rotation these two problems are, of course, virtually identical. Like the magnetic force of the present paper, the radial pressure gradient of the rotating gas resists the outflow behind the wave-front, and the disturbances there are damped in both cases. However, the analogy is far from complete; the decay of disturbances is like $r^{-\frac{1}{2}} e^{-kr}$ in the magnetic case, and like $r^{-\frac{1}{2}} e^{-k^* r^2}$ in the rotating-gas case (k^* being a constant, the details of which are irrelevant here), and the flow at substantial distances inside the wave-front is quite different in the two problems.

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Appendix. The orders of the electromagnetic variables

Let

$$\left.\begin{aligned} \mathbf{E} &= (\epsilon^q B_0 \sqrt{(gL)}\, e_1,\ \epsilon^q B_0 \sqrt{(gL)}\, e_2,\ \epsilon^l B_0 \sqrt{(gL)}\, e_3), \\ \mathbf{B} &= (\epsilon^m N B_0 b_1,\ \epsilon^m N B_0 b_2,\ B_0 + \epsilon^n N B_0 b_3). \end{aligned}\right\} \tag{A 1}$$

Our object in this Appendix is to select a set of values of q, l, m, n by a *reductio ad absurdum* of all schemes but one. Attention is confined to a liquid unbounded laterally. In view of conditions at the interfaces $X_3 = 0, H$, the orders q, m, n must have the same values inside and outside the liquid; l need not. We shall be concerned only with the lowest-order form of all equations; terms which, on the basis of what has gone before, cannot be among the dominant ones, will be neglected throughout. In each approximation of form $F = \epsilon^\nu f + \ldots$, $\epsilon^\nu f$ of course refers to the first non-vanishing term in the expansion of F. Hence any set of exponents (orders) which leads to any f vanishing everywhere, violates the definition of the ν and is inadmissible.

As before, we use the symbol α, which is 1 for $0 < X_3 < H$, and 0 for $X_3 < 0$ or $X_3 > H$: we shall also refer to these two domains by the words 'inside' and 'outside'. Maxwell's equations now become

$\nabla \wedge \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial T}$:

$$\left.\begin{aligned} \epsilon^{l}\frac{\partial e_3}{\partial x_2}-\epsilon^{q-\alpha}\frac{\partial e_2}{\partial x_3} &= -\epsilon^{m+\frac{1}{2}}N\frac{\partial b_1}{\partial t}, \\ \epsilon^{q-\alpha}\frac{\partial e_1}{\partial x_3}-\epsilon^{l}\frac{\partial e_3}{\partial x_1} &= -\epsilon^{m+\frac{1}{2}}N\frac{\partial b_2}{\partial t}, \\ \epsilon^{q}\left(\frac{\partial e_2}{\partial x_1}-\frac{\partial e_1}{\partial x_2}\right) &= -\epsilon^{n+\frac{1}{2}}N\frac{\partial b_3}{\partial t}. \end{aligned}\right\} \tag{A 2a, b, c}$$

$\nabla \wedge \mathbf{B} = \mu\sigma(\mathbf{E}+\mathbf{V}\wedge\mathbf{B})$:

$$\left.\begin{aligned} \epsilon^{n}\frac{\partial b_3}{\partial x_2}-\epsilon^{m-\alpha}\frac{\partial b_2}{\partial x_3} &= \alpha(\epsilon^{q}e_1+\epsilon^{\frac{1}{2}}v_2), \\ \epsilon^{m-\alpha}\frac{\partial b_1}{\partial x_3}-\epsilon^{n}\frac{\partial b_3}{\partial x_1} &= \alpha(\epsilon^{q}e_2-\epsilon^{\frac{1}{2}}v_1), \\ \epsilon^{m}\left(\frac{\partial b_2}{\partial x_1}-\frac{\partial b_1}{\partial x_2}\right) &= \alpha\,\epsilon^{l}e_3. \end{aligned}\right\} \tag{A 3a, b, c}$$

$\nabla . \mathbf{B} = 0$:

$$\epsilon^{m}\left(\frac{\partial b_1}{\partial x_1}+\frac{\partial b_2}{\partial x_2}\right)+\epsilon^{n-\alpha}\frac{\partial b_3}{\partial x_3} = 0. \tag{A 4}$$

(i) Assume that $n < m$. Then (A 4) reduces to $\partial b_3/\partial x_3 = 0$ inside and outside: b_3 vanishes at $|x_3| = \infty$ and is continuous across $x_3 = 0, h$, therefore $b_3 = 0$. Hence

$$m \leqslant n;$$

and in (A 3a, b), inside, $\epsilon^{n}\partial b_3/\partial x_2$, $\epsilon^{n}\partial b_3/\partial x_1$ are negligible.

(ii) Assume that $m < \frac{3}{2}$ or $q < \frac{1}{2}$. Then the velocity terms in (A 3a, b) are negligible, and Maxwell's equations and the boundary conditions are satisfied by $\mathbf{b} = 0$, $\mathbf{e} = 0$. Therefore $\quad \frac{3}{2} \leqslant m \leqslant n \quad$ and $\quad q \geqslant \frac{1}{2}$.

(iii) The vertical momentum equation now reduces to $0 = -\partial p/\partial x_3 - 1$ (provided that K is bounded as $\epsilon \to 0$), so that $p = h - x_3$, and the horizontal pressure gradient is independent of x_3.

(iv) In view of (ii) the horizontal current densities are now represented by

$$\epsilon^{q}e_1+\epsilon^{\frac{1}{2}}v_2, \quad \epsilon^{q}e_2-\epsilon^{\frac{1}{2}}v_1$$

(although the e_1, e_2 terms may be negligible), and the horizontal momentum equation becomes

$$\frac{Dv_1}{Dt} = -\frac{\partial h}{\partial x_1}+K\epsilon^{-\frac{1}{2}}(\epsilon^{q-\frac{1}{2}}e_2-v_1), \quad \frac{Dv_2}{Dt} = -\frac{\partial h}{\partial x_2}-K\epsilon^{-\frac{1}{2}}(\epsilon^{q-\frac{1}{2}}e_1+v_2). \tag{A 5a, b}$$

Hence we choose $K \sim O(\epsilon^{\frac{1}{2}})$, in order that the magnetic force shall not dominate the hydrostatic pressure gradient.

If $q > \frac{1}{2}$ the e-terms are negligible in (A 5); and if $q = \frac{1}{2}$, (A 2a, b) show, since $l > 0$, $m \geqslant \frac{3}{2}$, that $\partial e_1/\partial x_3 = 0$, $\partial e_2/\partial x_3 = 0$, inside. In either case, it follows from (A 5) that if (v_1, v_2) are independent of x_3 initially, as we assume, then they are so for all time.

(v) Assume that $q > \frac{1}{2}$. Then the magnetic force in (A 5) is proportional to the velocity. Rayleigh introduced such a term (see Lamb 1932, § 242) to represent a quasi-viscous force, and showed that under such a force irrotational flow is maintained, provided that the velocity is continuous; and continuous flows must be included in the approximate theory. For these, then

$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0.$$

Also, $m = \frac{3}{2}$, otherwise (A 3a, b) reduce to $v_1 = v_2 = 0$. (A 3a, b) can now be integrated to

$$b_1 = -v_1 x_3 + f_1(x_1, x_2), \quad b_2 = -v_2 x_3 + f_2(x_1, x_2),$$

and the boundary condition $\mathbf{n}.(\nabla \wedge \mathbf{B}) = 0$ on $X_3 = 0, H$, becomes

$$\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = 0 \quad (\text{on } X_3 = 0), \tag{A 6}$$

$$-\frac{\partial h}{\partial x_1} v_2 + \frac{\partial h}{\partial x_2} v_1 + \left(\frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2}\right) = 0 \quad (\text{on } X_3 = H). \tag{A 7}$$

Equations (A 5) ($q > \frac{1}{2}$), (A 6), and (A 7), all of which are independent of x_3, then combine to

$$v_2 \frac{Dv_1}{Dt} - v_1 \frac{Dv_2}{Dt} = 0, \quad \text{or} \quad \mathbf{i}_3 . \left(\mathbf{v} \wedge \frac{D\mathbf{v}}{Dt}\right) = 0. \tag{A 8}$$

But v_1 and v_2 are fully determined by the equations of continuity and momentum ($q > \frac{1}{2}$), and the boundary conditions; (A 8) is not, and cannot be, satisfied by a general flow with curved particle paths. Therefore

$$q = \tfrac{1}{2}.$$

(vi) Assume that $m > \frac{3}{2}$. (A 3a, b) reduce to

$$e_1 + v_2 = 0, \quad e_2 - v_1 = 0, \tag{A 9}$$

and the magnetic force vanishes from the hydrodynamic equations. Then if (v_1, v_2) are continuous and initially irrotational

$$\frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} = 0 \quad \text{and, by (A 9),} \quad \frac{\partial e_1}{\partial x_1} + \frac{\partial e_2}{\partial x_2} = 0.$$

Further by (A 2c), with $q = \frac{1}{2}$, $n > 0$,

$$\frac{\partial e_2}{\partial x_1} - \frac{\partial e_1}{\partial x_2} = 0.$$

Hence the vector (e_1, e_2) is harmonic, free of singularities and vanishes at infinity, so that $e_1 = e_2 = 0$. But this contradicts (A 9) (and the result $q = \frac{1}{2}$), hence

$$m = \tfrac{3}{2}.$$

(For plane or axi-symmetric flows, in which $q > \frac{1}{2}$, $m = \frac{3}{2}$ also, for otherwise (A 3a, b) reduce to $v_1 = v_2 = 0$.)

(vii) Assume that $n > \frac{3}{2}$. Then, outside, (A 3a, b) reduce to $\partial b_2/\partial x_3 = \partial b_1/\partial x_3 = 0$; b_1 and b_2 vanish at $|x_3| = \infty$, and are continuous across $x_3 = 0, h$; inside, $\partial b_1/\partial x_3$, $\partial b_2/\partial x_3$ are independent of x_3. Hence $b_1 = b_2 = 0$, and therefore

$$n = \tfrac{3}{2}.$$

(viii) It may be recalled that $l = l(\alpha)$. If $l(1) < \frac{3}{2}$, then by (A 3c) $e_3 = 0$ inside, which violates the definition of $l(1)$; if $l(1) > \frac{3}{2}$, then (A 3c) states that $(\partial b_2/\partial x_1) - (\partial b_1/\partial x_2) = 0$, in contradiction of results already established. Hence,

$$l(1) = \tfrac{3}{2}.$$

(ix) If $l(0) < \frac{1}{2}$, (A 2a, b) reduce to $\partial e_3/\partial x_1 = \partial e_3/\partial x_2 = 0$, and since e_3 vanishes for $x_1, x_2 \to \infty$, $e_3 = 0$ outside, which violates the definition of $l(0)$. If $l(0) > \frac{1}{2}$, (A 2a, b) lead to $e_1 = e_2 = 0$ outside, in contradiction of results already established. Hence,

$$l(0) = \tfrac{1}{2}.$$

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